

# DEFINING RELATIONS FOR LIE SUPERALGEBRAS WITH CARTAN MATRIX

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**ABSTRACT.** The notion of defining relations is well-defined for a nilpotent Lie (super)algebra. One of the ways to present a simple Lie algebra is, therefore, by splitting it into the direct sum of a maximal diagonalizing (commutative) subalgebra and 2 nilpotent subalgebras (positive and negative). The relations obtained for finite dimensional Lie algebras are neat; they are called *Serre relations* and can be encoded via an integer symmetrizable matrix, the *Cartan matrix*, which, in turn, can be encoded by means of a graph, the *Dynkin diagram*. The complete set of relations for Lie algebras with an arbitrary Cartan matrix is unknown.

We completely describe presentations of Lie superalgebras with Cartan matrix if they are simple  $\mathbb{Z}$ -graded of polynomial growth. Such matrices can be neither integer nor symmetrizable. There are *non-Serre* relations encountered. In certain cases there are infinitely many relations.

Our results are applicable to the Lie algebras with the same Cartan matrices as the Lie superalgebras considered.

## INTRODUCTION

This paper is the direct continuation of [K2], [LSS], [L1], [LSe]. In [LSe] the the case of the simplest (for computations) base is considered and non-Serre relations are first written. Though we are studying the Lie superalgebras with Cartan matrix, we give examples of Lie superalgebras of the other types, to illustrate the geometry related with some of our algebras.

An explicit presentation of simple Lie superalgebras became urgently needed in connection with  $q$ -quantization of Lie superalgebras: straightforward generalization of Drinfeld's results (who used Serre relations) is insufficient here, cf. [FLV]. After [FLV], there appeared a paper [Sch]  $q$ -quantizing  $\mathfrak{sl}(m|n)$  and, to an extent,  $\mathfrak{osp}(m|2n)$ . Though long, the paper [Sch] lacks explicit formulas for  $\mathfrak{osp}(m|2n)$  and even for  $\mathfrak{sl}(m|n)$  the explicit form of the general formulas is not given. Nowhere, so far, are the defining relations for all systems of simple roots of the exceptional Lie superalgebras written down (*some* systems are considered in [Y]).

Here for all simple Lie superalgebras  $\mathfrak{g}(A)$  with Cartan matrix  $A$  we list the defining relations for each system of simple roots. If  $\dim \mathfrak{g}(A) < \infty$  and for  $\mathbb{Z}$ -graded Lie superalgebras  $\mathfrak{g}(A)$  of polynomial growth with a symmetrizable  $A$  this list is complete (proof is the same as in [K1]); in the other cases its completeness is conjectured. Cartan matrices can be neither integer nor symmetrizable. There are *non-Serre* relations encountered. In certain cases there are infinitely many relations. We will consider the  $q$ -quantized versions elsewhere.

When Kac's method of the proof ([K1]) failed, and to derive our conjectures, we used the package [G]. To describe the defining relations for  $\mathfrak{g}(A)$  with symmetrizable matrices more general than those considered by Gaber and Kac (see [K1]) or nonsymmetrizable matrices distinct from those considered here is an open problem.

## §0. BACKGROUND

**0.0. Linear algebra in superspaces. Generalities.** Superization has certain subtleties, often disregarded or expressed too briefly, cf. [L]. We will dwell on them a bit.

A *superspace* is a  $\mathbb{Z}/2$ -graded space; for a superspace  $V = V_{\bar{0}} \oplus V_{\bar{1}}$  denote by  $\Pi(V)$  another copy of the same superspace: with the shifted parity, i.e.,  $(\Pi(V))_{\bar{i}} = V_{\bar{i}+\bar{1}}$ . The *superdimension* of  $V$  is  $\dim V = p + q\varepsilon$ , where  $\varepsilon^2 = 1$  and  $p = \dim V_{\bar{0}}$ ,  $q = \dim V_{\bar{1}}$ . (Usually,  $\dim V$  is expressed as a pair  $(p, q)$  or  $p|q$ ; this obscures the fact that  $\dim V \otimes W = \dim V \cdot \dim W$ .)

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A superspace structure in  $V$  induces the superspace structure in the space  $\text{End}(V)$ . A *superalgebra* is a superspace  $A$  with an even multiplication map  $m : A \otimes A \longrightarrow A$ .

A *basis of a superspace* is by definition a basis consisting of *homogeneous* vectors; let  $Par = (p_1, \dots, p_{\dim V})$  be an ordered collection of their parities. We call  $Par$  the *format* of  $V$ . A square *supermatrix* of format (size)  $Par$  is a  $\dim V \times \dim V$  matrix whose  $i$ th row and  $i$ th column are of the same parity  $p_i$ . The matrix unit  $E_{ij}$  is supposed to be of parity  $p_i + p_j$  and the bracket of supermatrices (of the same format) is defined via Sign Rule: *if something of parity  $p$  moves past something of parity  $q$  the sign  $(-1)^{pq}$  accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity*. For example: setting  $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$  we get the notion of the supercommutator and the ensuing notion of the Lie superalgebra (that satisfies the superskew-commutativity and super Jacobi identity).

We do not usually use the sign  $\wedge$  for differential forms on supermanifolds: in what follows we assume that the exterior differential is odd and the differential forms constitute a supercommutative superalgebra; we keep using it on manifolds, sometimes, not to deviate too far from the conventional notations.

Usually,  $Par$  is of the form  $(\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1})$ . Such a format is called *standard*. The nonstandard formats are vital in the classification of systems of simple roots; the corresponding defining relations are distinct.

**0.1.** The *general linear* Lie superalgebra of all supermatrices of size  $Par$  is denoted by  $\mathfrak{gl}(Par)$ , usually,  $\mathfrak{gl}(\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1})$  is abbreviated to  $\mathfrak{gl}(\dim V_0 | \dim V_1)$ . Any matrix from  $\mathfrak{gl}(m|n)$  (i.e., in the standard format) can be expressed as the sum of its even and odd parts:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad \text{where } p\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right) = \bar{0}, \quad p\left(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right) = \bar{1}.$$

More generally, we can consider matrices with the elements from a (usually, supercommutative) superalgebra  $C$ . Then the parity of the matrix with only one nonzero  $i, j$ -th entry  $X_{i,j} \in C$  is equal to  $p(i) + p(j) + p(X_{i,j})$ .

The *supertrace* is the map  $\mathfrak{gl}(Par) \longrightarrow \mathbb{C}$ ,  $(A_{ij}) \mapsto \sum (-1)^{p_i} A_{ii}$ . The superspace of supertraceless matrices constitutes the *special linear* Lie subsuperalgebra  $\mathfrak{sl}(Par)$ .

**Superalgebras that preserve bilinear forms: two types.** To the linear map  $F : V \longrightarrow W$  of superspaces there corresponds the dual map  $F^* : W^* \longrightarrow V^*$  of the dual superspaces; if  $A$  is the supermatrix corresponding to  $F$  in a format  $Par$ , then to  $F^*$  the *supertransposed* matrix  $A^{st}$  corresponds:

$$(A^{st})_{ij} = (-1)^{(p_i + p_j)(p_i + p(A))} A_{ji}.$$

The supermatrices  $X \in \mathfrak{gl}(Par)$  such that

$$X^{st}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for a homogeneous matrix } B \in \mathfrak{gl}(Par)$$

constitute the Lie superalgebra  $\mathfrak{aut}(B)$  that preserves the bilinear form on  $V$  with matrix  $B$ . If  $B$  corresponds to a nondegenerate supersymmetric form, then for the canonical form of its matrix in the standard format we can select either  $B_{ev}$  or  $B'_{ev}$  (each canonical form has its (dis)advantages):

$$B_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \quad \text{or } B'_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1, \dots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

The usual notation for  $\mathfrak{aut}(B_{ev}(m|2n))$  is  $\mathfrak{osp}^{sy}(m|2n)$ . (Observe that the passage from  $V$  to  $\Pi(V)$  sends the supersymmetric forms to superskew-symmetric ones, preserved by the “symplectico-orthogonal” Lie superalgebra  $\mathfrak{osp}^{sk}(m|2n)$  which is isomorphic to  $\mathfrak{osp}^{sy}(m|2n)$  but has a different matrix realization. We never use notation  $\mathfrak{sp}'\mathfrak{o}(2n|m)$  in order not to confuse with the special Poisson superalgebra. To understand this statement, recall that on the superspace of bilinear forms  $Bil(V)$  there is an involution  $u : B \mapsto B^u$  which on matrices acts as follows:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \mapsto B^u = \begin{pmatrix} B_{11}^t & B_{21}^t \\ B_{12}^t & B_{22}^t \end{pmatrix}.$$

In the standard format the matrix realizations of these algebras are:

$$\mathfrak{osp}^{sy}(m|2n) = \left\{ \begin{pmatrix} E & Y & -X^t \\ X & A & B \\ Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y^t & -X^t & E \end{pmatrix} \right\},$$

where  $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n)$ ,  $E \in \mathfrak{o}(m)$  and  $^t$  is the usual transposition.

A nondegenerate supersymmetric odd bilinear form  $B_{odd}(n|n)$  can be reduced to a canonical form whose matrix in the standard format is  $J_{2n}$ . A canonical form of the superskew odd nondegenerate form in the

standard format is  $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ . The usual notation for  $\mathfrak{aut}(B_{\text{odd}}(\text{Par}))$  is  $\mathfrak{pe}(\text{Par})$ . The passage from  $V$  to  $\Pi(V)$  sends the supersymmetric forms to superskew-symmetric ones and establishes an isomorphism  $\mathfrak{pe}^{sy}(\text{Par}) \cong \mathfrak{pe}^{sk}(\text{Par})$ . This Lie superalgebra is called, as A. Weil suggested, *periplectic*, i.e., odd-plectic. The matrix realizations in the standard format of these superalgebras is:

$$\begin{aligned} \mathfrak{pe}^{sy}(n) &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t, C = C^t \right\}; \\ \mathfrak{pe}^{sk}(n) &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t, C = -C^t \right\}. \end{aligned}$$

The *special periplectic* superalgebra is  $\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) : \text{str} X = 0\}$ .

**0.2. Vectoral Lie superalgebras. The standard realization.** The elements of the Lie algebra  $\mathcal{L} = \mathfrak{der} \mathbb{C}[[u]]$  are considered as vector fields. The Lie algebra  $\mathcal{L}$  has only one maximal subalgebra  $\mathcal{L}_0$  of finite codimension (consisting of the fields that vanish at the origin). The subalgebra  $\mathcal{L}_0$  determines a filtration of  $\mathcal{L}$ : set

$$\mathcal{L}_{-1} = \mathcal{L}; \quad \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} : [D, \mathcal{L}] \subset \mathcal{L}_{i-1}\} \text{ for } i \geq 1.$$

The associated graded Lie algebra  $L = \bigoplus_{i \geq -1} L_i$ , where  $L_i = \mathcal{L}_i / \mathcal{L}_{i+1}$ , consists of the vector fields with *polynomial* coefficients.

Unlike Lie algebras, simple vectoral *superalgebras* possess *several* maximal subalgebras of finite codimension.

**1) General algebras.** Let  $x = (u_1, \dots, u_n, \theta_1, \dots, \theta_m)$  the  $u_i$  are even indeterminates and the  $\theta_j$  are odd ones. The Lie superalgebra  $\mathfrak{vect}(n|m)$  is  $\mathfrak{der} \mathbb{C}[x]$ ; it is called *the general vectoral superalgebra*.

*Remark .* Sometimes we write  $\mathfrak{vect}(x)$  or even  $\mathfrak{vect}(V)$  if  $V = \text{Span}(x)$  and use similar notations for the subalgebras of  $\mathfrak{vect}$  introduced below. Algebraists sometimes abbreviate  $\mathfrak{vect}(n)$  and  $\mathfrak{svect}(n)$  to  $W_n$  (in honor of Witt) and  $S_n$ , respectively.

**2) Special algebras.** The *divergence* of the field  $D = \sum_i f_i \frac{\partial}{\partial u_i} + \sum_j g_j \frac{\partial}{\partial \theta_j}$  is the function (in our case: a polynomial, or a series)

$$\text{div} D = \sum_i \frac{\partial f_i}{\partial u_i} + \sum_j (-1)^{p(g_j)} \frac{\partial g_j}{\partial \theta_j}.$$

• The Lie superalgebra  $\mathfrak{svect}(n|m) = \{D \in \mathfrak{vect}(n|m) : \text{div} D = 0\}$  is called the *special* or *divergence-free vectoral superalgebra*.

It is clear that it is also possible to describe  $\mathfrak{svect}$  as  $\{D \in \mathfrak{vect}(n|m) : L_D \text{vol}_x = 0\}$ , where  $\text{vol}_x$  is the volume form with constant coefficients in coordinates  $x$  and  $L_D$  the Lie derivative with respect to  $D$ .

• The Lie superalgebra  $\mathfrak{svect}_\lambda(0|m) = \{D \in \mathfrak{vect}(0|m) : \text{div}(1 + \lambda \theta_1 \cdots \theta_m) D = 0\}$  — the deform of  $\mathfrak{svect}(0|m)$  — is called the *special* or *divergence-free vectoral superalgebra*. It is clear that  $\mathfrak{svect}_\lambda(0|m) \cong \mathfrak{svect}_\mu(0|m)$  for  $\lambda\mu \neq 0$ . Observe that  $p(\lambda) \equiv m \pmod{2}$ , i.e., for odd  $m$  the parameter of deformation  $\lambda$  is odd.

**3) The algebras that preserve Pfaff equations and differential 2-forms.** Set

$$\tilde{\alpha} = dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq m} \theta_j d\theta_j \quad \text{and} \quad \tilde{\omega} = d\tilde{\alpha}.$$

(Here we set  $u = (t, p_1, \dots, p_n, q_1, \dots, q_n)$ .) The form  $\alpha_1$  is called *contact*, the form  $\omega_0$  is called *symplectic*. Sometimes it is more convenient to redenote the  $\theta$ 's and set

$$\xi_j = \frac{1}{\sqrt{2}}(\theta_j - i\theta_{r+j}); \quad \eta_j = \frac{1}{\sqrt{2}}(\theta_j + i\theta_{r+j}) \quad \text{for } j \leq r = [m/2] \text{ (here } i^2 = -1), \quad \theta = \theta_{2r+1}$$

and in place of  $\tilde{\omega}$  or  $\tilde{\alpha}$  take  $\alpha$  and  $\omega = d\alpha$ , respectively, where

$$\begin{aligned} \alpha &= dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) & \text{if } m = 2r \\ \alpha &= dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) + \theta d\theta & \text{if } m = 2r + 1. \end{aligned}$$

The Lie superalgebra that preserves the *Pfaff equation*  $\alpha = 0$ , i.e., the superalgebra

$$\mathfrak{k}(2n+1|m) = \{D \in \mathfrak{vect}(2n+1|m) : L_D \alpha = f_D \alpha\},$$

(here  $f_D \in \mathbb{C}[t, p, q, \xi]$  is a polynomial determined by  $D$ ) is called the *contact superalgebra*. The Lie superalgebra

$$\mathfrak{po}(2n|m) = \{D \in \mathfrak{k}(2n+1|m) : L_D \alpha_1 = 0\}$$

is called the *Poisson superalgebra*. (A geometric interpretation of the Poisson superalgebra: it is the Lie superalgebra that preserves the connection with form  $\alpha$  in the line bundle over a symplectic supermanifold with the symplectic form  $d\alpha$ .)

**0.3. Generating functions.** A laconic way to describe  $\mathfrak{k}$  and its subalgebras is via generating functions.

Odd form  $\alpha$ . For  $f \in \mathbb{C}[t, p, q, \xi]$  set :

$$K_f = \Delta(f) \frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t} E,$$

where  $E = \sum_i y_i \frac{\partial}{\partial y_i}$  (here the  $y$  are all the coordinates except  $t$ ) is the *Euler operator* (which counts the degree with respect to the  $y$ ),  $\Delta(f) = 2f - E(f)$ , and  $H_f$  is the hamiltonian field with Hamiltonian  $f$  that preserves  $d\alpha_1$ :

$$H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \left( \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_j} \right), \quad f \in \mathbb{C}[p, q, \theta].$$

The choice of the form  $\alpha$  instead of  $\alpha$  only affects the form of  $H_f$  that we give for  $m = 2k + 1$ :

$$H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j \leq k} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right), \quad f \in \mathbb{C}[p, q, \xi, \eta, \theta].$$

Since

$$L_{K_f}(\alpha) = K_1(f) \cdot \alpha_1, \quad (0.1)$$

it follows that  $K_f \in \mathfrak{k}(2n+1|m)$ .

- To the supercommutator  $[K_f, K_g]$  there correspond *contact bracket* of the generating functions:

$$[K_f, K_g] = K_{\{f, g\}_{k.b.}}.$$

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on  $t$ .

The *Poisson bracket*  $\{\cdot, \cdot\}_{P.b.}$  (in the realization with the form  $\omega_0$ ) is given by the formula

$$\{f, g\}_{P.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j}$$

and in the realization with the form  $\omega'_0$  for  $m = 2k + 1$  it is given by the formula

$$\{f, g\}_{P.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - (-1)^{p(f)} \left[ \sum_{j \leq m} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j} \right) + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right].$$

Then

$$\{f, g\}_{k.b.} = \Delta(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} \Delta(g) - \{f, g\}_{P.b.}.$$

The Lie superalgebras of *Hamiltonian fields* (or *Hamiltonian superalgebra*) and its special subalgebra (defined only if  $n = 0$ ) are

$$\mathfrak{h}(2n|m) = \{D \in \mathfrak{vect}(2n|m) : L_D \omega_0 = 0\} \quad \text{and} \quad \mathfrak{sh}(m) = \{D \in \mathfrak{h}(0|m) : \text{div} D = 0\}.$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$\mathfrak{k}(2n+1|m) \cong \text{Span}(K_f : f \in \mathbb{C}[t, p, q, \xi]); \quad \mathfrak{h}(2n|m) \cong \text{Span}(H_f : f \in \mathbb{C}[p, q, \xi]).$$

*Remark .* 1) It is obvious that the Lie superalgebras of the series  $\mathfrak{vect}$ ,  $\mathfrak{svect}$ ,  $\mathfrak{h}$  and  $\mathfrak{po}$  for  $n = 0$  are finite dimensional.

2) A Lie superalgebra of the series  $\mathfrak{h}$  is the quotient of the Lie superalgebra  $\mathfrak{po}$  modulo the one-dimensional center  $\mathfrak{z}$  generated by constant functions. Similarly,  $\mathfrak{le}$  and  $\mathfrak{sle}$  are the quotients of  $\mathfrak{b}$  and  $\mathfrak{sb}$ , respectively, modulo the one-dimensional (odd) center  $\mathfrak{z}$  generated by constant functions.

3) There are analogues of the contact and hamiltonian series with an even 1-form,  $[L]$ .

Set  $\mathfrak{spo}(m) = \{K_f \in \mathfrak{po}(0|m) : \int f v_\xi = 0\}$ ; clearly,  $\mathfrak{sh}(m) = \mathfrak{spo}(m)/\mathfrak{z}$ .

**0.4. Nonstandard realizations.** In [LSh] we proved that the following are all the nonstandard gradings of the Lie superalgebras indicated. Moreover, the gradings in the series  $\mathbf{vect}$  induce the gradings in the series  $\mathbf{svect}$ , and  $\mathbf{svect}^\circ$ ; the gradings in  $\mathfrak{k}$  induce the gradings in  $\mathfrak{po}$ ,  $\mathfrak{h}$ . In what follows we consider  $\mathfrak{k}(2n+1|m)$  as preserving Pfaff eq.  $\alpha = 0$ . The standard realizations are marked by  $(*)$ ; note that (bar several exceptions for small  $m, n$ ) it corresponds to the case of the minimal codimension of  $\mathcal{L}_0$ . It corresponds to  $r = 0$ . There are also several exceptional nonstandard regradings; they are listed in sec. 2.6.

Lie superalgebra	its $\mathbb{Z}$ -grading
$\mathbf{vect}(n m; r),$ $0 \leq r \leq m$	$\deg u_i = \deg \xi_j = 1$ for any $i, j$ <span style="float: right;">(*)</span> $\deg \xi_j = 0$ for $1 \leq j \leq r$ ; $\deg u_i = \deg \xi_{r+s} = 1$ for any $i, s$
$\mathfrak{k}(2n+1 m; r), 0 \leq r \leq \lfloor \frac{m}{2} \rfloor$	$\deg t = 2, \deg p_i = \deg q_i = \deg \xi_j = \deg \eta_j = \deg \theta_k = 1$ for any $i, j, k$ <span style="float: right;">(*)</span> $\deg t = \deg \xi_i = 2, \deg \eta_i = 0$ for $1 \leq i \leq r \leq \lfloor \frac{m}{2} \rfloor$ ; $\deg p_i = \deg q_i = \deg \theta_j = 1$ for $j \geq 1$ and all $i$
$\mathfrak{k}(1 2m; m)$	$\deg t = \deg \xi_i = 1, \deg \eta_i = 0$ for $1 \leq i \leq m$

Observe that the Lie superalgebras corresponding to different values of  $r$  are isomorphic as abstract Lie superalgebras, but as filtered ones they are distinct.

**0.5. Stringy superalgebras.** These superalgebras are particular cases of the Lie algebras of vector fields, namely, those that preserve a structure on a what physicists call superstring, i.e., a supermanifold associated with a vector bundle on a circle. These superalgebras themselves are “stringy” indeed: as modules over the Witt algebra they are direct sums of several modules — strings.

Let  $\varphi$  be an angle parameter on a circle,  $t = \exp(i\varphi)$ . A *stringy superalgebra* is the algebra of derivations of either of the two supercommutative superalgebras

$$R^L(n) = \mathbb{C}[t^{-1}, t, \xi_1, \dots, \xi_n] \quad \text{or} \quad R^M(n) = \mathbb{C}[t^{-1}, t, \xi_1, \dots, \xi_{n-1}, \sqrt{t}\xi].$$

$R^L(n)$  is the superalgebra of complex-valued functions expandable into finite Fourier series or, as superscript indicates, Laurent series. These functions are considered of the real supermanifold  $S^{1|n}$  associated with the rank  $n$  trivial bundle over the circle. We can forget about  $\varphi$  and think in terms of  $t$  considered as the even coordinate on  $(\mathbb{C}^*)^{1|n}$ .

$R^M(n)$  is the superalgebra of complex-valued functions (expandable into finite Fourier series) on the supermanifold  $S^{1|n-1, M}$  associated with the Whitney sum of the Möbius bundle and the rank  $n-1$  trivial one. Since, as is well-known from Differential Geometry, the Whitney sum of two Möbius bundles is isomorphic to the trivial bundle of rank 2, it suffices to consider one Möbius summand.

Introduce analogues of  $\mathbf{vect}$ ,  $\mathbf{svect}$ ,  $\mathbf{svect}^\circ$  by substituting  $R^L(n)$  instead of  $R(n) = \mathbb{C}[t, \xi_1, \dots, \xi_n]$ :

$$\begin{aligned} \mathbf{vect}^L(n) &= \mathbf{der} R^L(n); \\ \mathbf{svect}_\lambda^L(n) &= \{D \in \mathbf{vect}^L(n) : \text{div}(t^\lambda D) = 0\} \\ \mathfrak{k}^L(n) &= \{D \in \mathbf{vect}^L(n) : D(\alpha_1) = f_D \alpha_1 \text{ for } \alpha_1 = dt + \sum \xi_i d\xi_i \text{ and } f_D \in R^L(n)\}. \end{aligned}$$

The same arguments as for  $\mathfrak{k}(2m+1|n)$ , prove that the elements that constitute  $\mathfrak{k}^L(n)$  are generated by functions, and the formula for  $K_f$  is the same as for  $\mathfrak{k}(1|n)$  with the only difference:  $f \in R^L(n)$ .

*Exercise .* The algebras  $\mathbf{vect}^M(n)$  and  $\mathbf{svect}_\lambda^M(n)$  obtained by replacing  $R(n)$  with  $R^M(n)$  are isomorphic to  $\mathbf{vect}^L(n)$  and  $\mathbf{svect}_{\lambda-\frac{1}{2}}^L(n)$ , respectively. Moreover,  $\mathbf{svect}_\lambda^L(n) \cong \mathbf{svect}_\mu^L(n)$  if and only if  $\lambda - \mu \in \mathbb{Z}$ .

If  $\lambda \in \mathbb{Z}$ , the Lie superalgebra  $\mathbf{svect}_\lambda^L(n)$  has a simple ideal of codimension  $\varepsilon^n$ :

$$0 \longrightarrow \mathbf{svect}_{\lambda}^{\circ L}(n) \longrightarrow \mathbf{svect}_\lambda^L(n) \longrightarrow \xi_1 \cdots \xi_n \partial_t \longrightarrow 0.$$

**0.6. Distinguished stringy superalgebras. Nontrivial central extensions.** Define the *residue* on  $S^{1|n}$  setting

$$\text{Res} : \text{Vol} \longrightarrow \mathbb{C}, \quad f \text{Vol}_{t, \xi} \mapsto \text{the coefficient of } \frac{\xi_1 \cdots \xi_n}{t} \text{ in the expansion of } f.$$

A simple Lie superalgebra is called *distinguished* if it has a nontrivial central extension. The following are all nontrivial central extensions:

algebra	cocycle	The name of the extended algebra
$\mathfrak{k}^L(1 0)$	$K_f, K_g \mapsto \text{Res} f K_1^3(g)$	Virasoro or $\mathfrak{vir}$
$\left. \begin{matrix} \mathfrak{k}^L(1 1) \\ \mathfrak{k}^M(1) \end{matrix} \right\}$	$K_f, K_g \mapsto \text{Res} f K_\theta(K_1)^2(g)$	Neveu-Schwarz or $\mathfrak{ns}$ Ramond or $\mathfrak{r}$
$\left. \begin{matrix} \mathfrak{k}^L(1 2) \\ \mathfrak{k}^M(2) \end{matrix} \right\}$	$K_f, K_g \mapsto \text{Res} f K_{\theta_1} K_{\theta_2} K_1(g)$	2-Neveu-Schwarz or $\mathfrak{ns}(2)$ 2-Ramond or $\mathfrak{r}(2)$
$\left. \begin{matrix} \mathfrak{k}^L(1 3) \\ \mathfrak{k}^M(3) \end{matrix} \right\}$	$K_f, K_g \mapsto \text{Res} f K_\xi K_\theta K_\eta(g)$	3-Neveu-Schwarz or $\mathfrak{ns}(3)$ 3-Ramond or $\mathfrak{r}(3)$
$\left. \begin{matrix} \mathfrak{k}^{L\circ}(4) \\ \mathfrak{k}^M(4) \end{matrix} \right\}$	$K_f, K_g \mapsto \begin{matrix} (1) & \text{Res} f K_{\theta_1} K_{\theta_2} K_{\theta_3} K_{\theta_4} (K_1)^{-1}(g) \\ (2) & \text{Res} f (t K_{t^{-1}}(g)) \\ (3) & \text{Res} f K_1(g) \end{matrix}$	(1) $\begin{cases} 4\text{-Neveu-Schwarz} = \mathfrak{ns}(4) \\ 4\text{-Ramond} = \mathfrak{ns}(4) \end{cases}$ (2) $\begin{cases} 4'\text{-Neveu-Schwarz} = \mathfrak{ns}(4') \\ 4'\text{-Ramond} = \mathfrak{ns}(4') \end{cases}$ (3) $\begin{cases} 4^0\text{-Neveu-Schwarz} = \mathfrak{ns}(4^0) \\ 4^0\text{-Ramond} = \mathfrak{ns}(4^0) \end{cases}$
$\mathfrak{vect}^L(1 2)$	the restrictions of the above cocycle (3) $D_1 = f \frac{\partial}{\partial t} + g_1 \frac{\partial}{\partial \xi_1} + g_2 \frac{\partial}{\partial \xi_2}$ $D_2 = \tilde{f} \frac{\partial}{\partial t} + \tilde{g}_1 \frac{\partial}{\partial \xi_1} + \tilde{g}_2 \frac{\partial}{\partial \xi_2}$ $\mapsto \text{Res}(g_1 \tilde{g}_2 - g_2 \tilde{g}_1 (-1)^{p(D_1)p(D_2)})$	$\widehat{\mathfrak{vect}}^L(1 2)$
$\mathfrak{svect}_\lambda^L(1 2)$	the restrictions of the above	$\widehat{\mathfrak{svect}}_\lambda^L(1 2)$

To see the formulas of the last two lines better, recall, that explicetly, the embedding  $\mathfrak{vect}(1|k) \longrightarrow \mathfrak{k}(1|2k)$  is given by the following formula in which  $\Phi = \sum \xi_i \eta_i$ :

$$\begin{aligned} f(\xi) x^n \partial_x &\mapsto (-1)^{p(f)} \frac{1}{2^n} f(\xi)(x + \Phi)^n \\ f(\xi) x^n \partial_i &\mapsto (-1)^{p(f)} \frac{1}{2^n} f(\xi) \eta_i (x + \Phi)^n \end{aligned}$$

Recall that  $\mathfrak{svect}_\lambda^L(1|2)$  is singled out by the formula

$$f \partial_x + \sum f_i \partial_i \in \mathfrak{svect}_\lambda^L(1|2) \quad \text{if and only if} \quad \lambda f = -x \text{div} D.$$

**0.7. Twisted loop superalgebras.** Let  $\mathfrak{g}$  be a simple finite dimensional Lie superalgebra,  $\varphi$  an automorphism of finite order  $k$ , let  $\varepsilon$  be a primitive root of 1 of degree  $k$ . The automorphism  $\varphi$  determines a  $\mathbb{Z}/k\mathbb{Z}$ -grading on  $\mathfrak{g}$  that we will denote by  $\mathfrak{g} = \bigoplus_{0 \leq i \leq k-1} \mathfrak{g}_i$ , where

$$\mathfrak{g}_i = \mathfrak{g}_i(\varphi) = \{g \in \mathfrak{g} : \varphi(g) = \varepsilon^i g\}.$$

The Lie superalgebra

$$\mathfrak{g}^{(1)} = \mathfrak{g}_{\text{id}}^{(1)} = \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t];$$

is called a *loop superalgebra*. The Lie superalgebra

$$\mathfrak{g}_\varphi^{(k)} = \bigoplus_{m \in \mathbb{Z}, 0 \leq j \leq k-1} \mathfrak{g}_j t^{mk+j}$$

is called a *twisted loop superalgebra*. (The maps of the circle somewhere are loops. So, the term “loop algebra” stems from the possibility to identify  $\mathfrak{g}^{(1)}$  and  $\mathfrak{g}_\varphi^{(k)}$  with the Lie superalgebra of  $\mathfrak{g}$ -valued functions on the circle expandable into finite Fourier series.)

In applications we encounter nontrivial central extensions of (twisted) loop superalgebras rather than the superalgebras themselves. The span of such an extension and the operator  $t \frac{d}{dt}$  will be called a *Kac–Moody superalgebra*.

**Theorem .** (Serganova, see [L], v. 22) a) *For a simple finite dimensional  $\mathfrak{g}$  and an automorphism  $\varphi \in \text{Out} \mathfrak{g}$  the superalgebra  $\mathfrak{g}_\varphi^{(k)}$  does not contain any nontrivial ideal homogeneous with respect to the  $\mathbb{Z}$ -grading defined by the formulas  $\deg g = 0$  for  $g \in \mathfrak{g}$ ,  $\deg t = 1$ .*

b) *Let  $\varphi_1$  and  $\varphi_2$  be two automorphisms of  $\mathfrak{g}$  of orders  $k_1$  and  $k_2$ , respectively. If  $\varphi_1 \varphi_2^{-1} \in \text{Aut}^0(\mathfrak{g})$ , then  $\mathfrak{g}_{\varphi_1}^{(k_1)} \cong \mathfrak{g}_{\varphi_2}^{(k_2)}$ .*

**0.8. Central extensions of (twisted) loop superalgebras.** There are two types of central extensions: one is associated with a nondegenerate supersymmetric invariant bilinear form on  $\mathfrak{g}$  (for the even form this is the straightforward generalization of the Kac–Moody cocycle), the other one is the series: functions with values in the extensions of  $\mathfrak{g}$ .

what is extended	the cocycle	The name of the result of the extension of $\mathfrak{g}_\varphi^{(m)}$
$\mathfrak{g}_\varphi^{(m)}$	1) $(X, Y) \mapsto \text{Res} B(X, \frac{dY}{dt})$ where $B$ is a symmetric bilinear form on $\mathfrak{g}$ 2) $c_i : X, Y \mapsto \text{Res} t^i c(X, Y)$ where $c$ is a nontrivial cocycle on $\mathfrak{g}$	$E_B(\mathfrak{g}_\varphi^{(m)})$ ,  $E_{c_i}(\mathfrak{g}_\varphi^{(m)})$ ,

**0.9. Exceptional algebras and  $\mathfrak{g}(A)$ .** All the Lie superalgebras described in this section are  $\mathbb{Z}$ -graded of polynomial growth, i.e., of the form  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ , where  $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$  (this means that  $\mathfrak{g}$  is graded);  $\dim \mathfrak{g}_i < \infty$  for all  $i$  and  $\sum_{|i| < n} \dim \mathfrak{g}_i$  grows as a polynomial in  $n$ . For the, so far limited, applications of the algebras whose growth is faster than a polynomial one see [K1].

Observe that the exceptional Lie algebras are rather difficult to describe. The same is true for Lie superalgebras. The only consise way to describe them is with the help of the Cartan matrix, i.e., to present them as  $\mathfrak{g}(A)$ .

### §1. WHAT IS $\mathfrak{g}(A)$ AND HOW TO PRESENT IT

First, recall, how to construct a Lie algebra from a Cartan matrix. Let  $A = (a_{ij})$  be an arbitrary complex  $n \times n$  matrix of rank  $l$ . Fix a complex vector space  $\mathfrak{h}$  of dimension  $2n - l$  and its dual  $\mathfrak{h}^*$ , select vectors  $h_1, h_2, \dots, h_n \in \mathfrak{h}$  and  $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$  so that  $\alpha_i(h_i) = a_{ij}$ .

Let  $I = \{i_1, \dots, i_n\} \subset (\mathbb{Z}/2\mathbb{Z})^n$ ; consider the free Lie superalgebra  $\tilde{\mathfrak{g}}(A, I)$  with generators  $e_1, \dots, e_n, f_1, \dots, f_n$  and  $h_1, h_2, \dots, h_n$ , where  $p(h_i) = \bar{0}$ ,  $p(e_j) = p(f_j) = i_j$  and defining relations:

$$[e_i, f_j] = \delta_{ij} h_j; [h, e_i] = \alpha_i(h) e_i; [h, f_i] = -\alpha_i(h) f_i; [\mathfrak{h}, \mathfrak{h}] = 0. \quad (1.1)$$

Let

$$Q = \sum_{1 \leq i \leq n} \mathbb{Z} \alpha_i; \quad Q^\pm = \{\alpha \in \mathfrak{h}^*; \alpha = \sum \pm n_i \alpha_i, n_i \in \mathbb{Z}_+\}.$$

For  $\alpha = \sum n_i \alpha_i \in Q$ , set  $ht(\alpha) = \sum n_i$ . We call  $Q$  the set *weights* and its subsets  $Q^\pm$  the sets of *positive* or *negative* weights, respectively;  $ht(\alpha)$  is the *height* of the weight  $\alpha$ .

**Statement .** ([K1], [vdL]) a) Let  $\tilde{\mathfrak{n}}_+$  and  $\tilde{\mathfrak{n}}_-$  be the superalgebras in  $\tilde{\mathfrak{g}}(A, I)$  generated by  $e_1, \dots, e_n$  and  $f_1, \dots, f_n$ , respectively; then  $\tilde{\mathfrak{n}}_+$  and  $\tilde{\mathfrak{n}}_-$  are free superalgebras with generators  $e_1, \dots, e_n$  and  $f_1, \dots, f_n$ , respectively, and  $\tilde{\mathfrak{g}}(A, I) \cong \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$ , as vector superspaces.

b) Among the ideals of  $\tilde{\mathfrak{g}}(A, I)$  with zero intersection with  $\mathfrak{h}$  there exists a maximal ideal  $\mathfrak{r}$  such that  $\mathfrak{r} = \mathfrak{r} \cap \tilde{\mathfrak{n}}_+ \oplus \mathfrak{r} \cap \tilde{\mathfrak{n}}_-$  is the direct sum of ideals.

Set  $\mathfrak{g}(A, I) = \tilde{\mathfrak{g}}(A, I)/\mathfrak{r}$ . Neither  $\mathfrak{g}(A, I)$  nor  $\mathfrak{g}(A, I)' = [\mathfrak{g}(A, I), \mathfrak{g}(A, I)]$  are simple. As proved in [vdL], the centers  $\mathfrak{c}$  of  $\mathfrak{g}(A, I)$  and  $\mathfrak{c}'$  of  $\mathfrak{g}(A, I)'$  consist of all  $h \in \mathfrak{h}$  such that  $\alpha_i(h) = 0$  for all  $i = 1, \dots, n$  and the quotient of  $\mathfrak{g}(A, I)'$  modulo the center is simple.

For the symmetrizable matrices  $A$  the simple Lie superalgebras of polynomial growth are listed in [vdL] (they are twisted loop superalgebras); for nonsymmetrizable ones Serganova proved (1989, unpublished) that these are only  $\mathfrak{psq}(n)^{(2)}$  and a stringy superalgebra  $\mathfrak{svect}_\alpha^L(1|2)$ . (Notice that there is a crucial difference between loop algebras and stringy algebras: in the former every root vector acts locally nilpotently ([K1]); this is false for the latter.)

Clearly, the rescaling  $(e_i \mapsto \sqrt{\lambda_i} e_i, f_i \mapsto \sqrt{\lambda_i} f_i)$  sends  $A$  to  $\text{diag}(\lambda_1, \dots, \lambda_n) \cdot A$ . Two pairs  $(A, I)$  and  $(A', I')$  are said to be *equivalent* if  $(A', I')$  is obtained from  $(A, I)$  by a permutation of indices or if  $A' = \text{diag}(\lambda_1, \dots, \lambda_n) \cdot A$ . Clearly, equivalent pairs determine isomorphic Lie superalgebra.

The matrix  $A$  (more precisely, a pair  $(A, I)$ ) is called a *Cartan matrix* of the Lie superalgebra  $\mathfrak{g}(A, I)$  and also of  $\tilde{\mathfrak{g}}(A, I)$ ,  $\mathfrak{g}'(A, I)$  as well as of  $\mathfrak{g}(A, I)/\mathfrak{c}$  and  $\mathfrak{g}'(A, I)/\mathfrak{c}'$ .

Let  $\mathfrak{g}$  be one of the Lie superalgebras  $\mathfrak{g}'(A, I)$ ,  $\mathfrak{g}(A, I)/\mathfrak{c}$  or  $\mathfrak{g}'(A, I)/\mathfrak{c}'$ . Set:

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} : [h, g] = \alpha(h)g \text{ for any } h \in \mathfrak{h}\}$$

and define the subalgebras  $\mathfrak{n}_\pm$  of  $\mathfrak{g}$  similarly to  $\tilde{\mathfrak{n}}_\pm$ .

A vector  $\alpha \in Q$  is called a *root* of  $\mathfrak{g}$  if  $\mathfrak{g}_\alpha \neq \{0\}$ . Denote by  $R$  the set of all the roots of  $\mathfrak{g}$  and let  $R^\pm = R \cap Q^\pm$ . In  $R$ , introduce a *parity* setting:

$$p(\alpha) = \sum n_j i_j, \text{ where } \alpha = \sum n_j \alpha_j \in R.$$

**Statement .** ([vdL]) Every Lie superalgebra  $\mathfrak{g}(A)$  possesses a root decomposition  $\mathfrak{g} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ , where  $\mathfrak{g}_0 = \mathfrak{h}$  and  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ .

Therefore, there exists a  $\mathbb{Z}$ -grading  $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ , where  $\mathfrak{g}_i = \bigoplus_{\alpha \in Q \text{ and } ht(\alpha)=i} \mathfrak{g}_\alpha$  with  $\mathfrak{n}_\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha$ .

**Corollary .**  $\mathfrak{h}$  is a maximal torus of  $\mathfrak{g}(A, I)$  and  $\tilde{\mathfrak{g}}(A, I)$  while  $\mathfrak{h}/\mathfrak{c}$  and  $\mathfrak{h}/\mathfrak{c}'$  are maximal tori of  $\mathfrak{g}(A, I)/\mathfrak{c}$  and  $\tilde{\mathfrak{g}}(A, I)/\mathfrak{c}'$ , respectively.

Our problem is to describe simple Lie superalgebras of the form  $\mathfrak{g}(A, I)$  more explicitly, i.e., to determine the generators of  $\mathfrak{r}$ . For an arbitrary  $A$  this is an open problem even for Lie algebras.

**1.1. Bases (systems of simple roots).** Let  $R$  be the root system of  $\mathfrak{g}$ . For any subset  $B = \{\sigma_1, \dots, \sigma_n\} \subset R$ , set:

$$R_B^\pm = \{\alpha \in R : \alpha = \pm \sum_{\pm} n_i \sigma_i, \quad n_i \in \mathbb{Z}_+\}.$$

Clearly,  $\dim \mathfrak{g}_{\pm \sigma_i} = (1, 0)$  or  $(0, 1)$  and  $R_B^+ \cap R_B^- = \{0\}$ .

The set  $B$  is called a *base* of  $R$  (or  $\mathfrak{g}$ ) or a *system of simple roots* if  $\sigma_1, \dots, \sigma_n$  are linearly independent and there exist  $\tilde{e}_1 \in \mathfrak{g}_{\sigma_1}, \dots, \tilde{e}_n \in \mathfrak{g}_{\sigma_n}, \tilde{f}_1 \in \mathfrak{g}_{-\sigma_1}, \dots, \tilde{f}_n \in \mathfrak{g}_{-\sigma_n}$  such that:

$$\mathfrak{g} = \oplus \mathfrak{g}_B^- \oplus \mathfrak{h} \oplus \mathfrak{g}_B^+,$$

where  $\mathfrak{g}_B^-$  (resp.  $\mathfrak{g}_B^+$ ) is the superalgebra generated by  $\tilde{e}_1, \dots, \tilde{e}_n$  (resp.  $\tilde{f}_1, \dots, \tilde{f}_n$ ).

Let  $B$  be a base and  $\tilde{e}_1, \dots, \tilde{e}_n, \tilde{f}_1, \dots, \tilde{f}_n$ , the corresponding elements of  $\mathfrak{g}$ . Set  $\tilde{h}_i = [\tilde{e}_i, \tilde{f}_i]$ ,  $A_B = (a_{ij})$ , where  $a_{ij} = \sigma_i(\tilde{h}_j)$  and  $I_B = \{p(\sigma_1), \dots, p(\sigma_n)\}$ .

The matrix  $A_B$  or, more precisely, the pair  $(A_B, I_B)$ , is called the *Cartan matrix* of  $\mathfrak{g}$ . The elements  $\tilde{e}_i, \tilde{f}_i$  and  $\tilde{h}_i$  for  $i = 1, \dots, n$ , satisfy the relations (1.1). Since there is no ideal in  $\mathfrak{g}$  with a non-zero intersection with  $\mathfrak{h}$ , we have:  $\mathfrak{g} = \mathfrak{g}(A, I)$ .

Two bases  $B_1$  and  $B_2$  are called *equivalent* if the pairs  $(A_{B_1}, I_{B_1})$  and  $(A_{B_2}, I_{B_2})$  are equivalent.

Hereafter  $\mathfrak{g} = \mathfrak{g}(A, I)$ . How many Cartan matrices correspond to the same Lie superalgebra  $\mathfrak{g}$ ? Let  $\mathfrak{g}(A)$  be a Lie superalgebra with Cartan matrix.

The following proposition due to V. Serganova lists, up to equivalence, all bases of  $\mathfrak{g}$  and, therefore, all Cartan matrices.

**Proposition .** *Let  $B$  be a base,  $\tilde{e}_i, \tilde{f}_i$ , for  $i = 1, \dots, n$  the corresponding set of generators and  $A_B = (a_{ij})$  the Cartan matrix. Fix an  $i$ . Then:*

- a) *If  $p(\sigma_i) = \bar{0}$  then, if  $\mathfrak{g}$  is of polynomial growth,  $a_{ii} \neq 0$  and the Lie subalgebra generated by the  $e_i$  and  $f_i$  is isomorphic to  $\mathfrak{sl}(2)$ .*
- b) *If  $p(\sigma_i) = \bar{1}$  and  $a_{ii} = 0$ , then  $\sigma_i \notin R$  and the subsuperalgebra generated by the  $e_i$  and  $f_i$  is isomorphic to  $\mathfrak{sl}(1|1)$ .*
- c) *If  $p(\sigma_i) = \bar{1}$  and  $a_{ii} \neq 0$ , then  $\sigma_i \notin R$  and the subsuperalgebra generated by the  $e_i$  and  $f_i$  is isomorphic to  $\mathfrak{osp}(1|2)$ .*

**1.3. Chevalley generators and odd reflections.** Let us multiply  $A_B$  from the right by a diagonal matrix so that in the cases a), b) or c) of Proposition 1.2 the diagonal elements of  $A_B$  become 2, 0 or 1, respectively. Such a matrix is said to be *normed*.

**Convention .** In what follows we only consider normed matrices.

A typical way to represent Lie algebras with integer Cartan matrices is via graphs called in the finite dimensional case Dynkin diagrams. The Cartan matrices of Lie superalgebras can be nonsymmetrizable or have complex entries; hence it is not always possible to assign to them an analog of the Dynkin diagram.

Every integer Cartan matrix  $(A, I)$  can be encoded with an analog of Dynkin diagram. Namely, the *Dynkin-Kac diagram* of the matrix  $(A, I)$  is the set of  $n$  nodes (vertices) connected by multiple edges, perhaps endowed with an arrow, according to the following rules. The nodes are of four types:

*To every simple root there corresponds*

$$\left\{ \begin{array}{ll} \text{a vertex } \circ & \text{if } p(\alpha_i) = \bar{0}, \text{ and } a_{ii} = 2 \\ \text{a vertex } \otimes & \text{if } p(\alpha_i) = \bar{1} \text{ and } a_{ii} = 0, \\ \text{a vertex } \bullet & \text{if } p(\alpha_i) = \bar{1} \text{ and } a_{ii} = 1; \\ \text{a vertex } * & \text{if } p(\alpha_i) = \bar{0}, \text{ and } a_{ii} = 0. \end{array} \right.$$

*A posteriori* we find out that the roots  $*$  can only occur if  $\mathfrak{g}(A, I)$  grows faster than polynomially.

Let the nodes corresponding to the  $i$ -th and the  $j$ -th roots be connected with  $\max(|a_{ij}|, |a_{ji}|)$  edges andowed with the sign  $>$  pointing to the  $j$ -th node if  $|a_{ij}| > |a_{ji}|$ .

• It turns out that an integer Cartan matrix  $(a_{ij})$  and a sequence  $I = \{i_1, \dots, i_n\}$  connected with a base can be uniquely, up to equivalence, recovered from their Dynkin diagram in all cases except  $\mathfrak{g} = \mathfrak{d}(\alpha)$ ,  $\mathfrak{g} = \mathfrak{d}(\alpha)^{(1)}$  or  $\mathfrak{sl}(2|4)^{(2)}$ . The procedure is as follows:

- 1) If the  $i$ -th and the  $j$ -th nodes are connected by  $k$  segments with an arrow pointing towards the  $j$ -th node, set:

$$|a_{ij}| = k, |a_{ji}| = 1;$$

- 2) If the  $i$ -th and the  $j$ -th nodes are joined by  $k$  segments without arrows, set

$$|a_{ij}| = |a_{ji}| = k.$$

- 3)
 

If the $j$ -th node is $\bullet$ , then	$a_{jj} = 1$	$i_j = \bar{1}$ ,
If the $j$ -th node is $\otimes$ , then	$a_{jj} = 0$	$i_j = \bar{1}$ ,
If the $j$ -th node is $\circ$ , then	$a_{jj} = 2$	$i_j = \bar{0}$ .



4) If  $a_{ii} \neq 0$  then  $a_{ji} \leq 0$  for any  $j \neq i$ .

5) If  $a_{ij} = 0$ , then the  $i$ -th column is recovered, up to multiplication by  $-1$ , as follows: if  $a_{ji} \neq 0$  for exactly one  $j$ , then the sign of  $a_{ji}$  may be chosen arbitrarily.

If  $a_{j_1 i}, a_{j_2 i} \neq 0$  for exactly two distinct indices  $j_1, j_2$ , then  $a_{j_1 i}/a_{j_2 i} < 0$ .

If  $a_{j_1 i}, a_{j_2 i}, a_{j_3 i} \neq 0$  for exactly three distinct indices  $j_1, j_2, j_3$  so that  $a_{j_1 j_2} \neq 0, a_{j_1 j_3} = a_{j_2 j_3} = 0$ , then  $a_{j_1 i}/a_{j_2 i} > 0, a_{j_1 i}/a_{j_3 i} < 0$ .

The set of generators corresponding to a normed matrix is often denoted in what follows by  $X_1^+, \dots, X_n^+$  and  $X_1^-, \dots, X_n^-$  instead of  $e_1, \dots, e_n$ , and  $f_1, \dots, f_n$ , respectively, and  $X_i^\pm, 1 \leq i \leq n$  are called the *Chevalley generators*.

The reflection in the  $i$ th root sends one set of Chevalley generators into the new one:  $\tilde{X}_i^\pm = X_i^\mp; \tilde{X}_j^\pm = [X_i^\pm, X_j^\pm]$  if  $a_{ij} \neq 0$  and  $\tilde{X}_j^\pm = X_j^\pm$  otherwise. The reflections in roots with  $a_{ii} \neq 0$  generate the Weyl group of  $\mathfrak{g}_0$ . For the discussion of what is generated by the other reflections, called *odd* ones, see [LSS], [S] and [E]. It is instructive to compare [E] with [PS].

**1.4. Serre-type relations.** Let  $\mathfrak{g} = \mathfrak{g}(A)$ . Let  $\mathfrak{h} \subset \mathfrak{g}$  be a maximal torus (i.e., the maximal diagonalizing commutative subalgebra),  $\mathfrak{g}^+, \mathfrak{g}^-$  subalgebras of  $\mathfrak{g}$  generated by root vectors corresponding to positive (resp. negative) roots; let the rank  $\text{rk } \mathfrak{g}$  be equal to  $n = \dim \mathfrak{g}$ ; let  $X_i^\pm$ , where  $1 \leq i \leq n$ , be root vectors corresponding to simple roots (for  $+$ ) and their opposite (for  $-$ ). Set  $H_i = [X_i^+, X_i^-]$ . It is subject to a direct verification that

$$[H_i, H_i] = 0, [X_i^+, X_j^-] = \delta_{ij} H_i, [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm \quad (SR_0)$$

Clearly, the generators of  $\mathfrak{n}^\pm$  are  $X_1^\pm, X_2^\pm, \dots, X_n^\pm$ . The defining relations are found by induction on  $n$  with the help of the Hochschild–Serre spectral sequence (see [Fu], [GM]). For the basis of the induction consider the following cases:

$$\circ \text{ or } \bullet : \text{ no relations; } \otimes : [X^\pm, X^\pm] = 0. \quad (1.4.1)$$

Set  $\deg X_i^\pm = 0$  for  $1 \leq i \leq n-1$  and  $\deg X_n^\pm = \pm 1$ . Let  $\mathfrak{n}^\pm = \oplus \mathfrak{n}_i^\pm, \mathfrak{g} = \oplus \mathfrak{g}_i$  be the corresponding  $\mathbb{Z}$ -gradings. From the Hochschild–Serre spectral sequence for the pair  $\mathfrak{n}_0^\pm \subset \mathfrak{n}^\pm$  we get (with  $\mathfrak{n}_\pm = \mathfrak{n}^\pm/\mathfrak{n}_0^\pm$ ):

$$H_2(\mathfrak{n}_0^\pm) \oplus H_1(\mathfrak{n}_0^\pm; H_1(\mathfrak{n}_\pm)) \oplus H_0(\mathfrak{n}_0^\pm; H_2(\mathfrak{n}_\pm)). \quad (1.4.2)$$

In the cases we are considering it is clear that

$$H_1(\mathfrak{n}_\pm) = \mathfrak{n}_1^\pm, \quad H_2(\mathfrak{n}_\pm) = E^2(\mathfrak{n}_1^\pm)/\mathfrak{n}_2^\pm \quad (1.4.3)$$

and, therefore, the second summand in (1.4.2) provides us with relations of the form:

$$(\text{ad } X_n^\pm)^{k_{ni}}(X_i^\pm) = 0 \quad \text{if the } n\text{-th root is not } \otimes$$

or

$$[X_n, X_n] = 0 \quad \text{if the } n\text{-th root is } \otimes.$$

while the third summand in (1.4.2) consists of  $\mathfrak{n}_0^\pm$ -lowest vectors in

$$E^2(\mathfrak{n}_1^\pm)/(\mathfrak{n}_2^\pm + \mathfrak{n}^\pm E^2(\mathfrak{n}_1^\pm)).$$

Let the matrix  $B = (b_{ij})$  be obtained from the Cartan matrix  $A = (a_{ij})$  by replacing all nonzero elements in the row with  $a_{ii} = 0$  by  $-1$  and multiplying the row with  $a_{ii} = 1$  by 2. The following proposition is straightforward:

**Proposition .** *The numbers  $k_{in}$  and  $k_{ni}$  are expressed in terms of  $(b_{ij})$  as follows:*

$$\begin{aligned} (\text{ad } X_i^\pm)^{1-b_{ij}}(X_j^\pm) &= 0 \quad \text{for } i \neq j \\ [X_i^\pm, X_i^\pm] &= 0 \quad \text{if } a_{ii} = 0 \end{aligned} \quad (SR_\pm)$$

The relations  $(SR_0)$  and  $(SR_\pm)$  will be called Serre relations for Lie superalgebra  $\mathfrak{g}(A)$ .

**1.5. Non-Serre-type relations.** Let us consider the simplest case:  $\mathfrak{sl}(m|n)$  in the realization with the base  $\bigcirc - \cdots - \bigcirc - \otimes - \cdots - \bigcirc$ . Then  $H_2(\mathfrak{n}_\pm)$  from the third summand in (2.2) is just  $E^2(\mathfrak{n}_\pm)$ .

Let us confine ourselves to the positive roots for simplicity. Let  $X_1, \dots, X_{m-1}; Y_1, \dots, Y_{n-1}$  be the root vectors corresponding to even roots,  $Z$  the root vector corresponding to the root  $\otimes$ .

If  $n = 1$  or  $m = 1$ , then  $E^2(\mathfrak{n})$  is an irreducible  $\mathfrak{n}_0$ -module and there are no non-Serre relations. If  $n \neq 1$  and  $m \neq 1$ , then  $E^2(\mathfrak{n})$  splits into 2 irreducible  $\mathfrak{n}_0$ -modules. The lowest component of one of them corresponds to the relation  $[Z, Z] = 0$ , the other one corresponds to the non-Serre-type relation

$$[[X_{m-1}, Z], [Y_1, Z]] = 0. \quad (*)$$

If instead of  $\mathfrak{sl}(m|n)$  we would have considered the Lie algebra  $\mathfrak{sl}(m+n)$  the same argument would have led us to the two relations:  $[Z, [Z, X_{m-1}]] = 0$  and  $[Z, [Z, Y_1]] = 0$  both of Serre type.

Let us consider the other root systems for the simplest example  $\mathfrak{sl}(1|n)$  to see what might happen. We start from the simplest base (one grey root) and apply to it odd reflections, see [PS], with respect to the first and then second root. We get the generators as indicated that satisfy, besides Serre relations, the relations indicated:

diagram	the corresponding generators	non-Serre relations
$\otimes - \circ - \circ - \circ - \circ -$	$X_1, X_2, X_3, X_4, X_5$	
$\otimes - \otimes - \circ - \circ - \circ -$	$X_1^-, [X_1, X_2], X_3, X_4, X_5$	$[[X_1, X_2], X_4] = 0$
$\circ - \otimes - \otimes - \circ - \circ -$	$X_2, [X_1^-, X_2^-], [[X_1, X_2], X_3], X_4, X_5$	$[[[X_1, X_2], X_3], X_2] = 0, [[X_1^-, X_2^-], X_4] = 0,$ $[[[X_1, X_2], X_3], X_5] = 0$

For  $\mathfrak{sl}(m+n)$  we similarly have (the passage from diagram to diagram is given by odd reflections in the 3rd, 4th, 2nd roots, respectively):

diagram	the corresponding generators	non-Serre relations
$\circ - \circ - \otimes - \circ - \circ -$	$X_1, X_2, X_3, X_4, X_5$	
$\circ - \otimes - \otimes - \otimes - \circ -$	$X_1, [X_3, X_2], X_3^-, [X_3, X_4], X_5$	$[[X_3, X_2], [X_3, X_4]] = 0$
$\circ - \otimes - \circ - \otimes - \otimes -$	$X_1, [X_3, X_2], X_4, [X_3^-, X_4^-],$ $[[X_3, X_4], X_5]$	$[[[X_1, X_2], X_3], X_2] = 0, \dots$
$\otimes - \otimes - \otimes - \otimes - \otimes -$	$[X_1, [X_3, X_2]], [X_3^-, X_2^-],$ $[[X_3, X_2], X_4], [X_3^-, X_4^-], [[X_3, X_4], X_5]$	$[[X_1, [X_2, X_3]], [[X_3, X_4], X_5]] = 0, \dots$

The idea of construction of the relations is clear, so we do not list all of them but leave the completion of the third column as an exercise. Contrarywise it is absolutely unclear, how to single out the *basic* relations. In what follows we list all the basic relations for the exceptional Lie superalgebras and for the “key cases” for the remaining series. The relations listed in [LSS] and [LSe], namely, all the Serre relations for *all* bases are, clearly, very redundant.

**Problem .** *How to single out the basic relations in the general case? How to describe the change of relations under the action of odd reflections?*

**1.6. Theorem .** *All the non-Serre-type relations become Serre relations after an appropriate odd reflection from a super Weyl group.* (For the definition of a super Weyl group see [LSS], [S] and [E].)

In Tables below for a finite dimensional algebra its dimension is indicated. The generators are assumed to correspond to positive roots. Their parities are determined by the corresponding diagonal elements of the Cartan matrix, since we do not consider the algebras of infinite growth. We only consider the Chevalley generators corresponding to the positive roots.

## §2. TABLE 2.1. RELATIONS FOR SYMMETRIZABLE CARTAN MATRICES

$\mathfrak{sl}(2|2)$ ,  $\dim = (7|8)$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_3] = 0 \\ [x_2, x_2] = 0 \\ [x_1, [x_1, x_2]] = 0 \\ [x_3, [x_2, x_3]] = 0 \\ [[x_1, x_2], [x_2, x_3]] = 0 \end{array} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_3] = 0 \\ [x_2, x_2] = 0 \\ [x_3, x_3] = 0 \\ [[x_1, x_2], [x_2, x_3]] = 0 \end{array}$$

$\mathfrak{sl}(1|3)$ ,  $\dim = (9|6)$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_3] = 0 \\ [x_2, [x_1, x_2]] = 0 \\ [x_2, [x_2, x_3]] = 0 \\ [x_3, [x_2, x_3]] = 0 \end{array} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_3] = 0 \\ [x_2, x_2] = 0 \\ [x_3, [x_2, x_3]] = 0 \\ [[x_1, x_2], [x_2, x_3]] = 0 \end{array}$$

-----  $\mathfrak{ag}_2$ ,  $\dim = (17|14)$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_3] = 0 \\ [x_2, [x_1, x_2]] = 0 \\ [x_3, [x_2, x_3]] = 0 \\ [x_2, [x_2, [x_2, [x_2, x_3]]]] = 0 \end{array} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_3] = 0 \\ [x_2, x_2] = 0 \\ [x_3, [x_2, x_3]] = 0 \\ [[x_1, x_2], [[x_1, x_2], [x_2, x_3]]]] = 0 \end{array}$$

$$\begin{pmatrix} 0 & -3 & 1 \\ -3 & 0 & 2 \\ -1 & -2 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_2, x_2] = 0 \\ [x_2, [x_1, x_3]] = -\frac{1}{3}[x_3, [x_1, x_2]] \\ [x_3, [x_1, x_3]] = 0 \\ [x_3, [x_3, [x_2, x_3]]] = 0 \end{array}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -3 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix} \quad \begin{aligned} [x_1, x_3] &= 0 \\ [x_2, x_2] &= 0 \\ [x_1, [x_1, x_2]] &= 0 \\ [[x_2, x_3], [x_3, x_3]] &= 0 \\ [x_3, [[x_1, x_2], [x_2, x_3]]] &= -\frac{1}{2}[[x_2, x_3], [x_3, [x_1, x_2]]] \end{aligned}$$


---

$\mathfrak{osp}(3|2)$ ,  $\dim = (6|6)$

$$\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_2, [x_2, [x_1, x_2]]] &= 0 \end{aligned} \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [[x_1, x_2], [x_2, x_2]] &= 0 \end{aligned}$$


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$\mathfrak{osp}(2|4)$ ,  $\dim = (11|8)$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_1, x_3] &= 0 \\ [x_2, [x_1, x_2]] &= 0 \\ [x_3, [x_2, x_3]] &= 0 \\ [x_2, [x_2, [x_2, x_3]]] &= 0 \end{aligned} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_1, x_3] &= 0 \\ [x_2, x_2] &= 0 \\ [x_3, [x_2, x_3]] &= 0 \\ [[x_1, x_2], [[x_1, x_2], [x_2, x_3]]] &= 0 \end{aligned}$$

$$\begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_2, x_2] &= 0 \\ [x_2, [x_1, x_3]] &= -\frac{1}{2}[x_3, [x_1, x_2]] \\ [x_3, [x_1, x_3]] &= 0 \\ [x_3, [x_2, x_3]] &= 0 \end{aligned}$$


---

$\mathfrak{osp}(3|4)$ ,  $\dim = (5|6)$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_3] &= 0 \\ [x_2, x_2] &= 0 \\ [x_1, [x_1, x_2]] &= 0 \\ [x_3, [x_3, [x_2, x_3]]] &= 0 \\ [[x_1, x_2], [x_2, x_3]] &= 0 \end{aligned} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_1, x_3] &= 0 \\ [x_2, x_2] &= 0 \\ [[x_1, x_2], [x_2, x_3]] &= 0 \\ [[x_2, x_3], [x_3, x_3]] &= 0 \end{aligned}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_1, x_3] &= 0 \\ [x_2, [x_1, x_2]] &= 0 \\ [x_2, [x_2, x_3]] &= 0 \\ [[x_2, x_3], [x_3, x_3]] &= 0 \end{aligned}$$


---

$\mathfrak{osp}(5|2)$ ,  $\dim = (13|10)$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \begin{aligned} [x_1, x_3] &= 0 \\ [x_2, x_2] &= 0 \\ [x_1, [x_1, x_2]] &= 0 \\ [[x_1, x_2], [x_2, x_3]] &= 0 \\ [[x_2, x_3], [x_3, x_3]] &= 0 \end{aligned} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_1, x_3] &= 0 \\ [x_2, x_2] &= 0 \\ [x_3, [x_3, [x_2, x_3]]] &= 0 \\ [[x_1, x_2], [x_2, x_3]] &= 0 \end{aligned}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_1, x_3] &= 0 \\ [x_2, [x_1, x_2]] &= 0 \\ [x_2, [x_2, x_3]] &= 0 \\ [x_3, [x_3, [x_2, x_3]]] &= 0 \end{aligned}$$


---

$\mathfrak{osp}_\alpha(4|2)$ ,  $\dim = (9|8)$

$$\begin{pmatrix} 2 & -1 & 0 \\ \alpha & 0 & -1-\alpha \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_3] &= 0 \\ [x_2, x_2] &= 0 \\ [x_1, [x_1, x_2]] &= 0 \\ [x_3, [x_2, x_3]] &= 0 \end{aligned} \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -\alpha \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{aligned} [x_1, x_3] &= 0 \\ [x_2, x_2] &= 0 \\ [x_1, [x_1, x_2]] &= 0 \\ [x_3, [x_2, x_3]] &= 0 \end{aligned}$$

$$\begin{pmatrix} 0 & 1 & -1-\alpha \\ -1 & 0 & -\alpha \\ -1-\alpha & \alpha & 0 \end{pmatrix} \quad \begin{aligned} [x_1, x_1] &= 0 \\ [x_2, x_2] &= 0 \\ [x_3, x_3] &= 0 \\ [x_2, [x_1, x_3]] &= (-1-\alpha)[x_3, [x_1, x_2]] \end{aligned}$$

$$\begin{pmatrix} 0 & 1 & -1-\alpha \\ -1 & 0 & -\alpha \\ -1-\alpha & \alpha & 0 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_2, x_2] = 0 \\ [x_3, x_3] = 0 \\ [x_2, [x_1, x_3]] = (-1-\alpha) [x_3, [x_1, x_2]] \end{array} \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1+\alpha \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_3] = 0 \\ [x_2, x_2] = 0 \\ [x_1, [x_1, x_2]] = 0 \\ [x_3, [x_2, x_3]] = 0 \end{array}$$


---

$\mathfrak{osp}(6|2)$ ,  $\dim = (18|12)$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_3] = 0 \\ [x_1, x_4] = 0 \\ [x_3, x_4] = 0 \\ [x_2, [x_1, x_2]] = 0 \\ [x_2, [x_2, x_3]] = 0 \\ [x_2, [x_2, x_4]] = 0 \\ [x_3, [x_2, x_3]] = 0 \\ [x_4, [x_2, x_4]] = 0 \end{array} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_3] = 0 \\ [x_1, x_4] = 0 \\ [x_2, x_2] = 0 \\ [x_3, x_4] = 0 \\ [x_3, [x_2, x_3]] = 0 \\ [x_4, [x_2, x_4]] = 0 \\ [[x_1, x_2], [x_2, x_3]] = 0 \\ [[x_1, x_2], [x_2, x_4]] = 0 \end{array}$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & -2 & 0 \end{pmatrix} \quad \begin{array}{l} [x_1, x_3] = 0 \\ [x_1, x_4] = 0 \\ [x_2, x_2] = 0 \\ [x_3, x_3] = 0 \\ [x_4, x_4] = 0 \\ [x_1, [x_1, x_2]] = 0 \\ [x_3, [x_2, x_4]] = [x_4, [x_2, x_3]] \\ [[x_1, x_2], [x_2, x_3]] = 0 \\ [[x_1, x_2], [x_2, x_4]] = 0 \end{array}$$


---

$\mathfrak{osp}(4|4)$ ,  $\dim = (16|16)$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_3] = 0 \\ [x_1, x_4] = 0 \\ [x_2, x_2] = 0 \\ [x_3, x_4] = 0 \\ [x_1, [x_1, x_2]] = 0 \\ [x_3, [x_2, x_3]] = 0 \\ [x_4, [x_2, x_4]] = 0 \\ [[x_1, x_2], [x_2, x_3]] = 0 \\ [[x_1, x_2], [x_2, x_4]] = 0 \end{array} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & -2 & 0 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_3] = 0 \\ [x_1, x_4] = 0 \\ [x_2, x_2] = 0 \\ [x_3, x_3] = 0 \\ [x_4, x_4] = 0 \\ [x_3, [x_2, x_4]] = [x_4, [x_2, x_3]] \\ [[x_1, x_2], [x_2, x_3]] = 0 \\ [[x_1, x_2], [x_2, x_4]] = 0 \end{array}$$


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$\mathfrak{ab}_3$ ,  $\dim = (24|16)$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_3] = 0 \\ [x_1, x_4] = 0 \\ [x_2, x_2] = 0 \\ [x_2, x_4] = 0 \\ [x_1, [x_1, x_2]] = 0 \\ [x_3, [x_2, x_3]] = 0 \\ [x_4, [x_3, x_4]] = 0 \\ [x_3, [x_3, [x_3, x_4]]] = 0 \\ [[x_3, x_4], [[x_1, x_2], [x_2, x_3]]] = 2 [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]] \end{array} \quad \begin{pmatrix} 0 & -3 & 1 & 0 \\ -3 & 0 & 2 & 0 \\ 1 & 2 & 0 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{l} [x_1, x_1] = 0 \\ [x_1, x_4] = 0 \\ [x_2, x_2] = 0 \\ [x_2, x_4] = 0 \\ [x_3, x_3] = 0 \\ [x_2, [x_1, x_3]] = -\frac{1}{3} [x_3, [x_1, x_2]] \\ [x_4, [x_3, x_4]] = 0 \\ [[x_2, x_3], [x_3, x_4]] = 0 \\ [[x_1, x_3], [[x_1, x_3], [x_3, x_4]]] = 0 \end{array}$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & -1 & 2 \end{pmatrix}
\begin{aligned}
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_2, x_4] = 0 \\
& [x_3, x_3] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_2, [x_1, x_2]] = 0 \\
& [x_2, [x_2, x_3]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [[[x_2, x_3], [x_3, x_4]], [[x_3, [x_1, x_2]], [[x_2, x_3], [x_3, x_4]]]] = 0
\end{aligned}$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 0 & 2 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}
\begin{aligned}
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_2, x_2] = 0 \\
& [x_3, x_3] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_3, [x_2, x_4]] = -\frac{1}{2}[x_4, [x_2, x_3]] \\
& [x_4, [x_2, x_4]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [[x_1, x_2], [x_2, x_3]] = 0
\end{aligned}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}
\begin{aligned}
& [x_1, x_1] = 0 \\
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_2, x_2] = 0 \\
& [x_2, x_4] = 0 \\
& [x_3, [x_2, x_3]] = 0 \\
& [x_3, [x_3, x_4]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [[x_1, x_2], [[x_1, x_2], [x_2, x_3]]] = 0
\end{aligned}$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\begin{aligned}
& [x_4, x_4] = 0 \\
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_2, x_4] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_2, [x_1, x_2]] = 0 \\
& [x_2, [x_2, x_3]] = 0 \\
& [x_3, [x_3, x_4]] = 0 \\
& [x_3, [x_3, [x_2, x_3]]] = 0
\end{aligned}$$

-----  
 $\mathfrak{ag}_2^{(1)}$  (relations computed up to degree 12)

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 4 & 0 & -1 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{pmatrix}
\begin{aligned}
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_2, x_2] = 0 \\
& [x_2, x_4] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_3, [x_2, x_3]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [x_3, [x_3, [x_3, [x_3, x_4]]]] = 0 \\
& [[[x_3, x_4], [[x_1, x_2], [x_2, x_3]]] = 3[[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]
\end{aligned}$$

$$\begin{pmatrix} 0 & -4 & 3 & 0 \\ -4 & 0 & 1 & 0 \\ 3 & 1 & 0 & -3 \\ 0 & 0 & -1 & 2 \end{pmatrix}
\begin{aligned}
& [x_1, x_1] = 0 \\
& [x_1, x_4] = 0 \\
& [x_2, x_2] = 0 \\
& [x_2, x_4] = 0 \\
& [x_3, x_3] = 0 \\
& [x_2, [x_1, x_3]] = -\frac{3}{4}[x_3, [x_1, x_2]] \\
& [x_4, [x_3, x_4]] = 0 \\
& [[x_1, x_3], [x_3, x_4]] = 0 \\
& [[x_2, x_3], [[x_2, x_3], [[x_2, x_3], [x_3, x_4]]]] = 0
\end{aligned}$$

$$\begin{array}{l}
\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -3 & 0 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix} \\
\begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 3 & -1 \\ 0 & 3 & 0 & -2 \\ 0 & -1 & -2 & 2 \end{pmatrix} \\
\begin{pmatrix} 1 & -1 & 0 & 0 \\ -2 & 0 & 3 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \\
\begin{pmatrix} 1 & -2 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}
\end{array}
\begin{array}{l}
[x_1, x_3] = 0 \\
[x_1, x_4] = 0 \\
[x_2, x_4] = 0 \\
[x_3, x_3] = 0 \\
[x_1, [x_1, x_2]] = 0 \\
[x_2, [x_1, x_2]] = 0 \\
[x_2, [x_2, x_3]] = 0 \\
[[x_3, x_4], [x_4, x_4]] = 0 \\
[x_4, [[x_2, x_3], [x_3, x_4]]] = -\frac{1}{2}[[x_3, x_4], [x_4, [x_2, x_3]]] \\
[x_1, x_3] = 0 \\
[x_1, x_4] = 0 \\
[x_2, x_2] = 0 \\
[x_3, x_3] = 0 \\
[x_1, [x_1, x_2]] = 0 \\
[x_3, [x_2, x_4]] = -\frac{1}{3}[x_4, [x_2, x_3]] \\
[x_4, [x_2, x_4]] = 0 \\
[x_4, [x_4, [x_3, x_4]]] = 0 \\
[[x_1, x_2], [x_2, x_3]] = 0 \\
[x_2, x_2] = 0 \\
[x_1, x_3] = 0 \\
[x_1, x_4] = 0 \\
[x_2, x_4] = 0 \\
[x_3, [x_2, x_3]] = 0 \\
[x_3, [x_3, x_4]] = 0 \\
[x_4, [x_3, x_4]] = 0 \\
[[x_1, x_1], [x_1, x_2]] = 0 \\
[[x_1, x_2], [x_3, [x_1, x_2]]] = -[[x_2, x_3], [x_2, [x_1, x_1]]] \\
[x_2, x_2] = 0 \\
[x_1, x_3] = 0 \\
[x_1, x_4] = 0 \\
[x_2, x_4] = 0 \\
[x_3, [x_2, x_3]] = 0 \\
[x_3, [x_3, x_4]] = 0 \\
[x_4, [x_3, x_4]] = 0 \\
[[x_1, x_1], [x_1, x_2]] = 0 \\
[[x_1, x_2], [x_3, [x_1, x_2]]] = -[[x_2, x_3], [x_2, [x_1, x_1]]]
\end{array}$$

-----  
 $\mathfrak{osp}(4|2)^{(2)}$  (relations computed up to degree 12)

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}
\begin{array}{l}
[x_1, x_3] = 0 \\
[x_2, x_2] = 0 \\
[x_3, [x_3, [x_2, x_3]]] = 0 \\
[[x_1, x_1], [x_1, x_2]] = 0 \\
[[x_1, x_2], [x_2, x_3]] = 0
\end{array}$$

-----  
 $\mathfrak{psl}(3|3)^{(4)}$  (relations computed up to degree 12)

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}
\begin{array}{l}
[x_1, x_3] = 0 \\
[x_2, x_2] = 0 \\
[x_1, [x_1, [x_1, x_2]]] = 0 \\
[x_3, [x_3, [x_2, x_3]]] = 0 \\
[[x_1, x_2], [x_2, x_3]] = 0 \\
[[x_1, [x_1, x_2]], [x_3, [x_2, x_3]]] = \frac{1}{2}[[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]]
\end{array}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}
\begin{array}{l}
[x_1, x_3] = 0 \\
[x_2, x_2] = 0 \\
[[x_1, x_1], [x_1, x_2]] = 0 \\
[[x_1, x_2], [x_2, x_3]] = 0 \\
[[x_2, x_3], [x_3, x_3]] = 0 \\
[[x_2, [x_1, x_1]], [x_3, [x_2, x_3]]] = -[[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]]
\end{array}$$

-----  
 $\mathfrak{osp}_\alpha(4|2)^{(1)}$ , (computed up to degree 12)

$$\begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -\alpha \\ 0 & 0 & 2 & 1+\alpha \\ -1 & -1 & -1 & 0 \end{pmatrix}
\begin{aligned}
& [x_4, x_4] = 0 \\
& [x_1, x_2] = 0 \\
& [x_1, x_3] = 0 \\
& [x_2, x_3] = 0 \\
& [x_1, [x_1, x_4]] = 0 \\
& [x_2, [x_2, x_4]] = 0 \\
& [x_3, [x_3, x_4]] = 0 \\
& [[x_2, x_4], [[x_1, x_4], [x_3, x_4]]] = -\frac{\alpha}{1+\alpha} [[x_3, x_4], [[x_1, x_4], [x_2, x_4]]]
\end{aligned}$$

$$\begin{pmatrix} 0 & -1 & -\alpha & 1+\alpha \\ -1 & 0 & 1+\alpha & -\alpha \\ -\alpha & 1+\alpha & 0 & -1 \\ 1+\alpha & -\alpha & -1 & 0 \end{pmatrix}
\begin{aligned}
& [x_1, x_1] = 0 \\
& [x_2, x_2] = 0 \\
& [x_3, x_3] = 0 \\
& [x_4, x_4] = 0 \\
& [x_2, [x_1, x_3]] = \alpha [x_3, [x_1, x_2]] \\
& [x_2, [x_1, x_4]] = -(1+\alpha) [x_4, [x_1, x_2]] \\
& [x_3, [x_1, x_4]] = -\frac{1+\alpha}{\alpha} [x_4, [x_1, x_3]] \\
& [x_3, [x_2, x_4]] = -\frac{\alpha}{1+\alpha} [x_4, [x_2, x_3]]
\end{aligned}$$

-----  
 $\mathfrak{ab}_3^{(1)}$  (computed up to degree 12)

$$\begin{pmatrix} 2 & -3 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}
\begin{aligned}
& [x_2, x_2] = 0 \\
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_1, x_5] = 0 \\
& [x_2, x_4] = 0 \\
& [x_2, x_5] = 0 \\
& [x_3, x_5] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_3, [x_2, x_3]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [x_4, [x_4, x_5]] = 0 \\
& [x_5, [x_4, x_5]] = 0 \\
& [x_3, [x_3, [x_3, x_4]]] = 0 \\
& [[x_3, x_4], [[x_1, x_2], [x_2, x_3]]] = 2 [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]
\end{aligned}$$

$$\begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -2 & 0 & 2 \end{pmatrix}
\begin{aligned}
& [x_2, x_2] = 0 \\
& [x_3, x_3] = 0 \\
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_1, x_5] = 0 \\
& [x_2, x_5] = 0 \\
& [x_4, x_5] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_3, [x_2, x_4]] = -\frac{1}{2} [x_4, [x_2, x_3]] \\
& [x_4, [x_2, x_4]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [x_5, [x_3, x_5]] = 0 \\
& [[x_1, x_2], [x_2, x_3]] = 0 \\
& [[x_2, x_3], [x_3, x_5]] = 0
\end{aligned}$$

$$\begin{pmatrix} 0 & -3 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}
\begin{aligned}
& [x_1, x_1] = 0 \\
& [x_2, x_2] = 0 \\
& [x_3, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_1, x_5] = 0 \\
& [x_2, x_4] = 0 \\
& [x_2, x_5] = 0 \\
& [x_3, x_5] = 0 \\
& [x_2, [x_1, x_3]] = -\frac{1}{3} [x_3, [x_1, x_2]] \\
& [x_4, [x_3, x_4]] = 0 \\
& [x_4, [x_4, x_5]] = 0 \\
& [x_5, [x_4, x_5]] = 0 \\
& [[x_2, x_3], [x_3, x_4]] = 0 \\
& [[x_1, x_3], [[x_1, x_3], [x_3, x_4]]] = 0 \\
& [x_2, x_2] = 0 \\
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_1, x_5] = 0 \\
& [x_2, x_4] = 0 \\
& [x_2, x_5] = 0 \\
& [x_3, x_5] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_3, [x_2, x_3]] = 0 \\
& [x_3, [x_3, x_4]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [x_4, [x_4, x_5]] = 0 \\
& [x_5, [x_4, x_5]] = 0 \\
& [[[x_1, x_2], [x_2, x_3]], [x_4, [x_2, x_3]], [[x_1, x_2], [x_2, x_3]]] = 0
\end{aligned}$$


---

$\mathfrak{sl}(2|4)^{(2)}$ , (computed up to degree 21)

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}
\begin{aligned}
& [x_2, x_2] = 0 \\
& [x_3, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_1, [x_1, x_3]] = 0 \\
& [x_2, [x_1, x_3]] = [x_3, [x_1, x_2]] \\
& [x_3, [x_2, x_4]] = -\frac{1}{2} [x_4, [x_2, x_3]] \\
& [x_4, [x_2, x_4]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [x_2, x_2] = 0 \\
& [x_3, x_3] = 0 \\
& [x_4, x_4] = 0 \\
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_3, [x_2, x_4]] = [x_4, [x_2, x_3]] \\
& [[[x_2, x_3], [[x_1, x_2], [x_2, x_3]]] = 0 \\
& [[x_2, x_4], [[x_1, x_2], [x_2, x_4]]] = 0 \\
& [x_3, x_3] = 0 \\
& [x_1, x_3] = 0 \\
& [x_1, x_4] = 0 \\
& [x_2, x_4] = 0 \\
& [x_1, [x_1, x_2]] = 0 \\
& [x_2, [x_2, x_3]] = 0 \\
& [x_4, [x_3, x_4]] = 0 \\
& [x_2, [x_2, [x_1, x_2]]] = 0 \\
& [[[x_2, x_3], [x_3, x_4]], [[x_3, x_4], [[x_1, x_2], [x_2, x_3]]]] = \\
& \quad - [[[x_2, x_3], [x_3, x_4]], [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]]
\end{aligned}$$



TABLE 2.2. RELATIONS FOR NONSYMMETRIZABLE CARTAN MATRICES

There are two series of such algebras:  $\mathfrak{svect}_\alpha^L(1|2)$  and  $\mathfrak{psq}(n)^{(2)}$ .  
 $\mathfrak{psq}(3)^{(2)}$  (computed up to degree 21)

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}
 \begin{aligned}
 & [x_2, x_2] = 0 \\
 & [x_1, [x_1, x_2]] = 0 \\
 & [x_1, [x_1, x_3]] = 0 \\
 & [x_3, [x_1, x_3]] = 0 \\
 & [x_3, [x_2, x_3]] = 0 \\
 & [[x_2, [x_1, x_3]], [x_2, [x_1, x_3]]] = \\
 & \quad [[x_2, [x_1, x_3]], [x_3, [x_1, x_2]]] - [[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]] \\
 & \quad [[x_1, [[x_1, x_2], [x_2, x_3]]], [[x_2, [x_1, x_3]], [x_3, [x_3, [x_1, x_2]]]] = \\
 & \quad -6 [[x_1, x_3], [x_3, [x_1, x_2]]], [[x_3, [x_1, x_2]], [[x_1, x_2], [x_2, x_3]]] \\
 & \quad -2 [[x_2, x_3], [x_3, [x_1, x_2]]], [[x_3, [x_1, x_2]], [x_1, [x_3, [x_1, x_2]]]] \\
 & [x_1, x_1] = 0 \\
 & [x_2, x_2] = 0 \\
 & [x_3, x_3] = 0 \\
 & [[x_1, x_2], [x_2, [x_1, x_3]]] = -\frac{1}{2} [[x_1, x_2], [x_3, [x_1, x_2]]] \\
 & [[x_1, x_3], [x_2, [x_1, x_3]]] = -2 [[x_1, x_3], [x_3, [x_1, x_2]]] \\
 & [[x_2, x_3], [x_2, [x_1, x_3]]] = [[x_2, x_3], [x_3, [x_1, x_2]]] \\
 & [[x_1, x_3], [[x_1, x_2], [x_2, x_3]]] = \frac{1}{2} [[x_2, x_3], [[x_1, x_2], [x_1, x_3]]] + \frac{3}{4} [[x_2, [x_1, x_3]], [x_3, [x_1, x_2]]] \\
 & \quad [[[x_1, x_2], [x_3, [x_1, x_2]]], [[x_3, [x_1, x_2]], [[x_1, x_3], [x_2, x_3]]]] = \\
 & \quad -4 [[x_2, x_3], [x_3, [x_1, x_2]]], [[x_3, [x_1, x_2]], [[x_1, x_2], [x_1, x_3]]] \\
 & \quad + 4 [[x_1, x_3], [x_3, [x_1, x_2]]], [[x_3, [x_1, x_2]], [[x_1, x_2], [x_2, x_3]]] \\
 & \quad [[[[x_1, x_2], [x_1, x_3]], [[x_1, x_3], [x_2, x_3]]], [[[x_1, x_2], [x_3, [x_1, x_2]]], [[x_2, x_3], [x_3, [x_1, x_2]]]] = \\
 & \quad -\frac{3}{4} [[[[x_1, x_2], [x_2, x_3]], [[x_1, x_3], [x_3, [x_1, x_2]]]], [[[x_1, x_3], [x_2, x_3]], [[x_1, x_2], [x_3, [x_1, x_2]]]]] \\
 & \quad -\frac{1}{2} [[[[x_1, x_2], [x_2, x_3]], [[x_1, x_3], [x_2, x_3]]], [[[x_1, x_2], [x_3, [x_1, x_2]]], [[x_1, x_3], [x_3, [x_1, x_2]]]]]
 \end{aligned}$$

-----  
 $\mathfrak{psq}(4)^{(2)}$ , (computed up to degree 21)

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}
 \begin{aligned}
 & [x_4, x_4] = 0 \\
 & [x_1, x_3] = 0 \\
 & [x_2, x_4] = 0 \\
 & [x_1, [x_1, x_2]] = 0 \\
 & [x_1, [x_1, x_4]] = 0 \\
 & [x_2, [x_1, x_2]] = 0 \\
 & [x_2, [x_2, x_3]] = 0 \\
 & [x_3, [x_2, x_3]] = 0 \\
 & [x_3, [x_3, x_4]] = 0 \\
 & [[x_1, x_4], [x_3, x_4]] = 0 \\
 & [[x_3, [x_1, x_4]], [[x_3, x_4], [x_4, [x_1, x_2]]]] = 0 \\
 & [[x_4, [x_1, x_2]], [[x_3, x_4], [x_3, [x_1, x_2]]]] = [[x_4, [x_2, x_3]], [[x_1, x_4], [x_3, [x_1, x_2]]]] \\
 & [[x_3, [x_1, x_2]], [[x_1, x_4], [x_2, x_3]]], [[[x_1, x_4], [x_2, x_3]], [[x_3, x_4], [x_4, [x_1, x_2]]]] = \\
 & \quad \frac{1}{2} [[[[x_4, [x_1, x_2]], [[x_1, x_4], [x_2, x_3]]], [[[x_1, x_4], [x_2, x_3]], [[x_3, x_4], [x_3, [x_1, x_2]]]]] \\
 & \quad + \frac{1}{2} [[[[x_4, [x_2, x_3]], [[x_1, x_4], [x_2, x_3]]], [[[x_1, x_4], [x_2, x_3]], [[x_1, x_4], [x_3, [x_1, x_2]]]]] \\
 & [x_1, x_1] = 0 \\
 & [x_3, x_3] = 0 \\
 & [x_4, x_4] = 0 \\
 & [x_1, x_3] = 0 \\
 & [x_2, x_4] = 0 \\
 & [x_2, [x_1, x_2]] = 0 \\
 & [x_2, [x_2, x_3]] = 0 \\
 & [[x_1, x_2], [x_1, x_4]] = 0 \\
 & [[x_1, x_4], [x_3, x_4]] = 0 \\
 & [[x_2, x_3], [x_3, x_4]] = 0 \\
 & [[x_3, [x_1, x_2]], [x_3, [x_1, x_4]]] = 0 \\
 & [[x_4, [x_1, x_2]], [[x_3, x_4], [x_3, [x_1, x_2]]]] = \\
 & \quad [[x_4, [x_3, [x_1, x_2]]], [[x_1, x_4], [x_2, x_3]]] - [[x_4, [x_2, x_3]], [[x_1, x_4], [x_3, [x_1, x_2]]]] \\
 & \quad - [[x_4, [x_3, [x_1, x_2]]], [[x_1, x_2], [x_3, x_4]]] \\
 & \quad [[x_4, [x_1, x_2]], [[x_1, x_4], [x_2, x_3]]], [[[x_1, x_4], [x_2, x_3]], [[x_3, x_4], [x_3, [x_1, x_2]]]] = \\
 & \quad [[x_4, [x_2, x_3]], [[x_1, x_4], [x_2, x_3]]], [[[x_1, x_4], [x_2, x_3]], [[x_1, x_4], [x_3, [x_1, x_2]]]]
 \end{aligned}$$

-----  
 $\mathbf{svect}_\alpha^L(1|2)$ , (computed up to degree 12)

$$\begin{pmatrix} 2 & -1 & -1 \\ 1-\alpha & 0 & \alpha \\ 1+\alpha & -\alpha & 0 \end{pmatrix} \quad \begin{aligned} [x_2, x_2] &= 0 \\ [x_3, x_3] &= 0 \\ [x_1, [x_1, x_2]] &= 0 \\ [x_1, [x_1, x_3]] &= 0 \\ [[x_2, x_3], [x_2, [x_1, x_3]]] &= \frac{\alpha+1}{\alpha-1} [[x_2, x_3], [x_3, [x_1, x_2]]] \end{aligned}$$

### §3. COMMENTS

**3.0.** The relations of Tables 2.1 and 2.2 are the defining ones in all the cases except  $\mathfrak{psl}(3|3)^{(4)}$  and  $\mathfrak{psq}(3)^{(2)}$ . In the last two cases there are *infinitely* many relations that kill the cocycles  $c_i$ , see sec. 0.8. The degrees of these relations grow with  $i$ . (Though these relations look awful when expressed in terms of the Chevalley generators, they are easy to describe in terms of the matrix units, cf. 0.8.)

Conjectorally, these are the only relations additional to the listed ones.

**3.1. Statement .** (On  $3 \times 3$  matrices) *There is a relation between  $[x_1, [x_2, x_3]]$  and  $[x_2, [x_3, x_1]]$  for any symmetric Cartan matrix  $(A_{ij})$  with*

$$A_{12} + A_{13} + A_{23} = 0.$$

*The relation does not depend on the diagonal elements  $A_{ii}$ . The relation exists for any parity of generators and is not reducible to the Serre relations if  $A_{12}A_{13}A_{23} \neq 0$ . This relation, together with the Jacobi identity for  $x_1, x_2$  and  $x_3$ , may be written as*

$$\frac{(-1)^{P(x_1)P(x_3)}}{A_{23}} [x_1, [x_2, x_3]] = \frac{(-1)^{P(x_1)P(x_2)}}{A_{13}} [x_2, [x_3, x_1]] = \frac{(-1)^{P(x_2)P(x_3)}}{A_{12}} [x_3, [x_1, x_2]].$$

*For a nonsymmetrizable matrix such that  $(A_{ij} = 0) \iff (A_{ji} = 0)$ , this relation is impossible.*

*If  $x_2$  is odd,  $A_{13} = A_{22} = A_{31} = 0$  and  $A_{23} = -pA_{21}$ , then the relation  $(ad_{[x_1, x_2]})^P([x_2, x_3]) = 0$  holds.*

**3.2. Remark .** It seems that if the ratio  $A_{31} : A_{23}$  is a negative rational, but neither integer nor the inverse of an integer, there is one more relation.

**3.3. Statement .** (On  $4 \times 4$  matrices) *For the Cartan matrix (with anything instead of each  $*$ )*

$$\begin{pmatrix} * & 1 & p & -pq \\ 1 & * & pq & -p \\ p & pq & * & -1 \\ -pq & -p & -1 & * \end{pmatrix}$$

*there is a relation between  $[[x_1, x_2], [x_3, x_4]]$ ,  $[[x_1, x_3], [x_2, x_4]]$  and  $[[x_1, x_4], [x_2, x_3]]$ , namely:*

$$pq [[x_1, x_2], [x_3, x_4]] - (-1)^{P(x_2)P(x_3)} q [[x_1, x_3], [x_2, x_4]] - (-1)^{(P(x_2)+P(x_3))P(x_4)} [[x_1, x_4], [x_2, x_3]] = 0.$$

**3.4. Two statements on  $n \times n$  matrices ( $n \geq 4$ ).** Serre relations involve just two generators. We have seen that even for  $\mathfrak{sl}(m|n)$  there are relations involving 5 generators. It seems that ge

1) Let  $\deg x_j = 1$  for all  $j$ ; set  $y_j = x_j^-$ . In the free Lie algebra generated by the  $x_j$  denote by  $v_i$  the expressions of degree 1 with respect to *each*  $x_j$ . Comparing the number of equations  $[y_k, \sum c_j v_j] = 0$  with the number of parameters ( $A_{ij}$  and  $c_j$ ), we see that there exist relations of degree  $n$  involving all the  $x_1, \dots, x_n$  for  $n \leq 5$ . We cannot say more about the case  $n \geq 6$ .

2) If  $\mathfrak{g}$  has a central extension, so  $\mathfrak{g}_\phi^{(m)}$  has infinitely many central extensions, and, consequently, infinitely many defining relations: each central element of positive degree has to be equated to zero. Examples of such relations are the last indicated relations for  $\mathfrak{psl}(3|3)^{(4)}$  and  $\mathfrak{psq}(3)^{(2)}$ .

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